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Self-consistent localization theory in magnetic fields and the upper critical field of disordered superconductors

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Abstract. We revisit the theory of the upper critical field of disordered superconductors and we present an alternative calculation of B_{c_2} where we avoid semiclassical arguments and work from the start with Landau levels. As in other current theories we neglect Coulomb interactions and concentrate on localization effects, and this we do by generalizing the self-consistent theory by Vollhardt and Wölfle to a system lacking translational and time reversal invariance. We analyse the theory in configuration space and discuss the approximations that lead to closed self-consistent equations that would reproduce well known results in the non-magnetic case. In the low-field limit we obtain a pair of coupled equations for the frequency-renormalized diffusion coefficient and conductivity. As a result of our detailed calculation from first principles we detect a suppression of the diffusion pole in one of the equations that was overlooked in a phenomenological theory. The results for the upper critical field exhibit a change from negative to positive curvature in the B_{c_2} versus T plot for increasing disorder and they are compared with those obtained in the theory of Kotliar and Kapitulnik.

1. Introduction

The classical theory of disordered superconductors in the presence of a magnetic field was formulated by Werthamer, Helfand and Hohenberg [1], but recent developments in localization theory [2] together with new experimental results [3,4] have led theorists to generalize these early calculations. In particular, the experimental evidence presented in [3] reveals a pronounced decrease in the critical-field slope parameter of superconducting thin films as a function of increasing normal-state sheet resistance, which results in a change from negative to positive curvature in the B_2 versus T plot, and it is interpreted as a disorder-induced localization effect [2]. Speaking loosely, one can see how the conductivity affects the critical-field behaviour by expanding the implicit solution in the classical theory [1] for $T \leq T_c$ which gives $B_{c_2} \approx (T_c - T)/D_0$ where $D_0 = \frac{1}{3}\epsilon_F \tau$ is the diffusion coefficient. One may expect that when localization effects are taken into account the former expression would be modified by replacing $D_0 \to D_0 \Delta$ where $\Delta = \sigma(T)/\sigma_0$ would be the ratio of the localization-renormalized conductivity to the Drude value σ_0.

At the metal insulator transition one would expect that $\sigma(T)$ would vanish on decreasing T with some algebraic behaviour, $\sigma(T) \approx (T/T_F)^\delta$, which would produce an automatic increase of $B_{c_2}(T)$ above its classical value for low temperatures. Another

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line of thought, however, is that localization effects in superconductors appear through the renormalization of the Coulomb repulsion and the phonon-induced attraction [4]. In spite of these ideas looking simple and appealing, a rigorous microscopic calculation is difficult and involved as it requires a full use of diagrammatic many-body techniques to solve self-consistently for the superconducting properties in the presence of a metal–insulator transition, with the extra complication that the superconducting electrons are no longer free but move in a magnetic field that acts as a pair-breaker in the superconductor, but also destroys particle–hole symmetry and reduces localization. Several authors concentrated on the effects that the disorder-renormalized Coulomb interaction produces on the superconducting properties [5–7], and the results in [6] give an enhancement of B_{c_2} at low temperatures while keeping always a negative curvature in the B_{c_2} versus T plot.

Along a different line of approach, Kotliar and Kapitulnik [8] presented a theory that neglects Coulomb interactions while strong-disorder effects are incorporated into the scale dependence of the diffusion coefficient. They obtain as a result a positive curvature in the B_{c_2} versus T plot in agreement with the experiments. Their method of calculation, like all previous work, is based on the standard 'semiclassical approximation' that consists in the following: the superconducting kernel $K(q)$ is first evaluated at zero magnetic field by exploiting translational invariance and particle–hole symmetry, in terms of the diffusion coefficient that had been previously calculated in localization theory [9].

The magnetic field dependence is incorporated phenomenologically afterwards by writing $K(q = \sqrt{\omega_c^*})$ in place of $K(q = 0)$ in the equation for the transition temperature, where $\omega_c^* = 2eB/mc$ is the cyclotron frequency for a doubly charged particle. Equivalent ways of obtaining the semiclassical results are, either by approximating the true electron Green function by the Green function at zero field times a phase factor, or by replacing $\frac{1}{2m} q_\perp^2 \rightarrow \omega_c^*(n + \frac{1}{2})$, where $q_\perp$ is the magnitude of the momentum transfer on the plane normal to the magnetic field.

In the present work we revisit the problem and we present an alternative calculation of B_{c_2} from first principles, where we avoid the semiclassical arguments and work from the start with the true electron eigenstates in a magnetic field, as was previously done for the magnetoconductivity [10]. Although the semiclassical approach is expected to give good results in the low-field, strong-disorder limit $\omega_c \tau < 1$, its asymptotic character does not permit the calculation of systematic corrections. For instance, the results in [10] show anisotropic effects in the conductivity tensor that have been overlooked in the semiclassical calculations. We consider the same model as in [8], which neglects Coulomb interactions, and localization effects are introduced by generalizing the self-consistent theory of Vollhardt and Wölfle [11] to a system lacking translational invariance and particle–hole symmetry, which requires some technical discussions.

The superconducting kernel is obtained from the particle–particle reducible vertex part by attaching four propagators to it; then we write first the Bethe–Salpeter equations for the particle–particle and the particle–hole reducible vertex parts in the approximation that considers only diagonal matrix elements [12], which gives us a pair of tractable equations. We obtain in this way an integral equation for the superconducting kernel and we calculate exactly the lowest eigenvalue that determines the transition temperature, in terms of the particle–particle irreducible vertex that according to [11] is obtained from the particle–hole reducible vertex part.

The integral equation for the particle–hole vertex part turns out to be translationally invariant even in the presence of the field due to the cancellation of phases in the propagators going in opposite directions; then it is solved by Fourier transformation. However, in the corresponding integral equation for the particle–particle vertex part the phases of the propagators going in the same direction add and an explicit solution could only be obtained to lowest order in the field [10].

This paper is organized as follows: in section 2 we derive the equation for the critical field in the low-field limit in terms of a renormalized diffusion coefficient $D_0\Delta_\nu$, where $D_0 = \frac{1}{3}\epsilon_F\tau$ is the diffusion constant and Δ_ν is a frequency-dependent parameter inversely proportional to the particle–particle irreducible vertex part. In section 3 we derive the equation for Δ_ν by applying the self-consistent localization theory [11], which produces a pair of coupled equations for Δ_ν and the frequency-dependent conductivity that we indicate by σ_ν. The explicit expressions obtained in the low-field limit are compared with the results by Yoshioka *et al* [13] for the magnetoconductivity, and we obtained an additional suppresion of the diffusion pole due to the difference in vertex parts that was overlooked in [13].

In section 4 we discuss the approximate solutions for Δ_ν in limiting cases and the corresponding results for B_{c_2}.

The expected change in curvature is obtained when we consider the weak-field, strong-disorder regime as shown in figure 4, later.

2. Upper critical field

We consider a system of superconducting electrons in the presence of a uniform magnetic field $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ in the z-direction and of a random impurity potential. At the critical point the gap vanishes and relevant physical properties are obtained by calculating the configurational average of products of Green functions that are the solutions of the equation:

$$\left[\pm \mathrm{i}\omega_\nu + \epsilon_F + \tfrac{1}{2}\,\boldsymbol{D}_r^2 - V(\boldsymbol{r})\right] \tilde{G}_{\pm\nu}(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}') \tag{1}$$

where $\omega_\nu = (2\nu+1)\,\pi/\beta$ and

$$\boldsymbol{D}_r = \boldsymbol{\nabla}_r - \mathrm{i}e\,\boldsymbol{A}(\boldsymbol{r}). \tag{2}$$

We choose to work in the gauge $\boldsymbol{A} = (-By, 0, 0)$ and $V(\boldsymbol{r})$ is the impurity potential with zero mean and variance:

$$\langle V(\boldsymbol{r})\,V(\boldsymbol{r}')\rangle = U\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{3}$$

The equation for the upper critical field $B_{c_2}(T)$ is [1]:

$$\frac{\lambda}{\beta}\sum_\nu S_\nu = 1 \tag{4}$$

where $\lambda > 0$ is the strength of the attractive BSC interaction while S_ν is the lowest eigenvalue of the integral equation:

$$\int \mathrm{d}\boldsymbol{r}'\, K_\nu(\boldsymbol{r},\boldsymbol{r}')\,\Delta(\boldsymbol{r}') = S_\nu\,\Delta(\boldsymbol{r}) \tag{5}$$

with the kernel:

$$K_\nu(\boldsymbol{r},\boldsymbol{r}') = \langle \tilde{G}_\nu(\boldsymbol{r},\boldsymbol{r}')\,\tilde{G}_{-\nu}(\boldsymbol{r},\boldsymbol{r}')\rangle \tag{6}$$

where the brackets indicate an average over random impurities as in (3).

The impurity-averaged quantities are obtained through standard diagrammatic methods, only now the diagrams should be written in configuration space because of the lack of translational invariance.

The averaged one-particle Green function satisfies the integral equation:

$$G_\nu(\boldsymbol{r},\boldsymbol{r}') = G_\nu^0(\boldsymbol{r},\boldsymbol{r}') + \int \mathrm{d}\boldsymbol{r}_1\,\mathrm{d}\boldsymbol{r}_2\,G_\nu^0(\boldsymbol{r},\boldsymbol{r}_1)\Sigma_\nu(\boldsymbol{r}_1,\boldsymbol{r}_2)G_\nu(\boldsymbol{r}_2,\boldsymbol{r}') \tag{7}$$

where some of the diagrams contributing to the self-energy $\Sigma_\nu(\boldsymbol{r},\boldsymbol{r}')$ are shown in figure 1(a) and $G_\nu^0(\boldsymbol{r},\boldsymbol{r}')$ is the solution of (1) for $V=0$.

The kernel in (5) is shown schematically in figure 1(b) as a function of the reducible particle–particle vertex part, $W_{\nu\mathrm{R}}^{\mathrm{p-p}}$, which satisfies the integral equation of figure 2(a). In order to apply the self-consistent localization theory of Vollhardt and Wölfle [11], we should consider that the particle–particle vertex part, $W_\nu^{\mathrm{p-p}}$, is directly obtained from the reducible particle–hole vertex, $W_{\nu\mathrm{R}}^{\mathrm{p-h}}$, which is shown in figure 2(b).

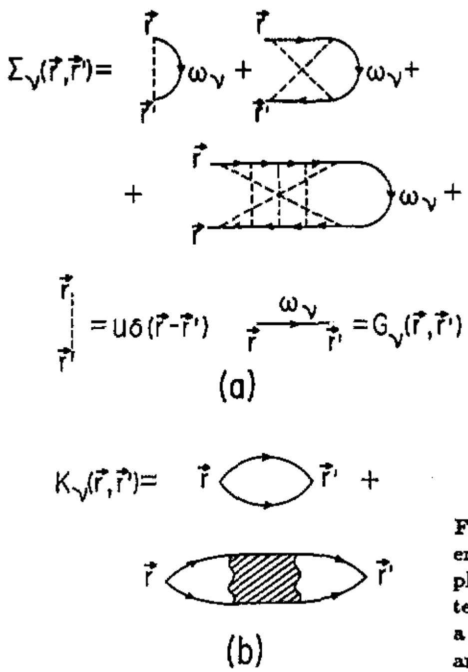

Figure 1. (a) Some diagrams contributing to the self-energy operator $\Sigma_\nu(\boldsymbol{r},\boldsymbol{r}')$. Full lines indicate a complete propagator $G_\nu(\boldsymbol{r},\boldsymbol{r}')$, pointed lines the bare interactions $U\delta(\boldsymbol{r}-\boldsymbol{r}')$. (b) Superconducting kernel as a function of the particle–particle vertex part (hatched area).

We proceed to evaluate these formal expressions in the approximation that considers only the contribution of diagonal matrix elements [12].

If we approximate [12]

$$W_\nu^{\mathrm{p-p}}(\boldsymbol{r}_1\boldsymbol{r}_2\mid\boldsymbol{r}_3\boldsymbol{r}_4) \approx \Pi_\nu^{\mathrm{p-p}}\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\,\delta(\boldsymbol{r}_3-\boldsymbol{r}_4)\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_3) \tag{8}$$

in figure 2(a) we obtain that

$$W_{\nu R}^{p-p}(\boldsymbol{r}_1\boldsymbol{r}_2 \mid \boldsymbol{r}_3\boldsymbol{r}_4) \approx W_{\nu R}^{p-p}(\boldsymbol{r}_1, \boldsymbol{r}_2)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_3)\,\delta(\boldsymbol{r}_2 - \boldsymbol{r}_4) \tag{9}$$

while for the kernel in figure 1(b) we get the integral equation:

$$K_\nu(\boldsymbol{r}, \boldsymbol{r}') = K_\nu^0(\boldsymbol{r}, \boldsymbol{r}') + \Pi_\nu^{p-p} \int \mathrm{d}\boldsymbol{r}_1\; K(\boldsymbol{r}, \boldsymbol{r}_1)\; K^0(\boldsymbol{r}_1, \boldsymbol{r}') \tag{10}$$

where:

$$K_\nu^0(\boldsymbol{r}, \boldsymbol{r}') = G_\nu(\boldsymbol{r}, \boldsymbol{r}')\; G_{-\nu}(\boldsymbol{r}, \boldsymbol{r}'). \tag{11}$$

The self-energy is related to the irreducible particle–particle vertex part by the Ward identity:

$$\mathrm{Im}\,\Sigma_\nu(\boldsymbol{r}, \boldsymbol{r}') = \int \mathrm{d}\boldsymbol{r}_1\, \mathrm{d}\boldsymbol{r}_2\; W_\nu^{p-p}(\boldsymbol{r}\boldsymbol{r}_1 \mid \boldsymbol{r}_2\boldsymbol{r}')\,\mathrm{Im}\,G_\nu(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{12}$$

which can be proved by generalizing the method of Maleev *et al* [14] to non-translational invariant systems. Some diagrams for W_ν^{p-p} are shown in figure 3.

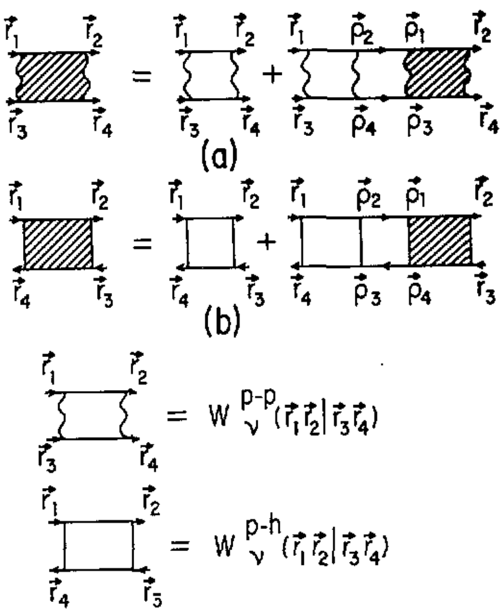

Figure 2. Bethe–Salpeter equation for (a) the reducible particle–particle vertex function $W_{\nu R}^{p-p}$ and (b) the reducible particle–hole vertex function $W_{\nu R}^{p-h}$.

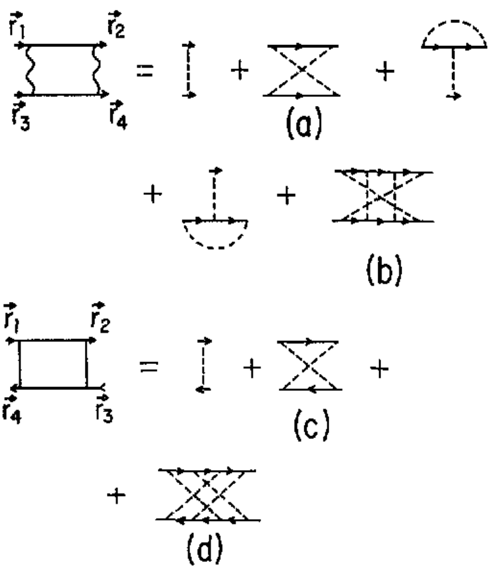

Figure 3. Some diagrams contributing to the irreducible particle–particle vertex function W_ν^{p-p} (top) and to the irreducible particle–hole function W_ν^{p-h} (bottom).

In the same approximation we should write:

$$\Sigma_\nu(\boldsymbol{r},\boldsymbol{r}') \approx \mathrm{i}\,\Gamma_\nu\,\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{13}$$

and we obtain from (12) together with (8) the simplified identity:

$$\Gamma_\nu = \Pi_\nu^{\mathrm{p-p}}\,\mathrm{Im}\,G_\nu(\boldsymbol{r},\boldsymbol{r}). \tag{14}$$

By using (13) the average propagator can be calculated as in [10] to obtain:

$$G_\nu(\boldsymbol{r},\boldsymbol{r}') = g_\nu(\boldsymbol{r}-\boldsymbol{r}')\ \exp\left(\frac{\mathrm{i}\omega_c}{2}\,(y+y')(x-x')\right). \tag{15}$$

Hence $G_\nu(\boldsymbol{r},\boldsymbol{r})$ is position independent as is required for the consistency of (14). We have in (15):

$$g_\nu(\boldsymbol{r}) = \frac{\omega_c}{2\pi}\sum_{n=0}^{\infty}\int\frac{\mathrm{d}q}{2\pi}\ \exp\left(-\mathrm{i}qz-\frac{\omega_c}{4}\,\rho^2\right)L_n\left(\frac{\omega_c}{2}\,\rho^2\right)\,G_\nu(n,p). \tag{16}$$

where $\boldsymbol{\rho}=(x,y)$ is the position vector on the plane normal to the field, $L_n(z)$ indicates a Laguerre polynomial, $\omega_c = eB$ and:

$$G_\nu(n,p) = \left[\omega_c\left(n+\frac{1}{2}\right)+\frac{1}{2}\,p^2-\epsilon_{\mathrm{F}}-\mathrm{i}(\omega_\nu+\Gamma_\nu)\right]^{-1}. \tag{17}$$

It has been shown before [1] that the eigenvalue of (5) with the kernel of (10) is given by:

$$S_\nu = \frac{S_\nu^0}{1-\Pi_\nu^{\mathrm{p-p}}\,S_\nu^0} \tag{18}$$

where S_ν^0 satisfies:

$$\int\mathrm{d}\boldsymbol{r}'\ K_\nu^0(\boldsymbol{r},\boldsymbol{r}')\,\Delta(\boldsymbol{r}') = S_\nu^0\,\Delta(\boldsymbol{r}). \tag{19}$$

By introducing (11) and (15) into (19) we obtain the exact solution:

$$\Delta(\boldsymbol{r}) = \Delta_0\ \exp\left(\mathrm{i}\omega_c\,xy-\frac{\omega_c}{2}\,\rho^2\right) \tag{20}$$

with

$$S_\nu^0 = \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\mathrm{d}q\,\omega_c\sum_{M=0}^{\infty}2^{-M}\sum_{n=0}^{M}\binom{M}{n}\mathrm{Im}\left\{\left[\omega_\nu+\Gamma_\nu+\mathrm{i}\omega_c\left(n-\frac{M}{2}\right)\right]^{-1}G_\nu(n,q)\right\}. \tag{21}$$

The standard result is obtained from (21) in an expansion to lowest order in ω_c, which gives:

$$S_\nu^0 \approx \frac{\rho_{\mathrm{F}}}{|\omega_\nu+\Gamma_\nu|}\ \left(1-\frac{1}{3}\,\frac{\omega_c\,\epsilon_F}{(\omega_\nu+\Gamma_\nu)^2}\right) \tag{22}$$

where

$$\rho_{\rm F} = {\rm Im}\, G_\nu(0) = \frac{\sqrt{2\epsilon_{\rm F}}}{2\pi}. \tag{23}$$

We get by introducing (22) into (18) and using (14):

$$S_\nu \approx \rho_{\rm F} \left(|\,\omega_\nu\,| + \frac{1}{3}\,\omega_{\rm c}\,\epsilon_{\rm F}\,\frac{|\,\Gamma_\nu\,|}{(\omega_\nu + \Gamma_\nu)^2} \right)^{-1} \tag{24}$$

which reduces to the classical result [1] if we approximate $\Gamma_\nu \approx \rho_{\rm F}\,U$, while to take into account localization effects one should include higher-order contributions to $\Pi_\nu^{\rm p-p}$ in (14), as is discussed in the next section.

From (24) and (4) we obtain the implicit equation for the critical field, in the disordered limit when Γ_ν is much larger than the Debye frequency:

$$\ln(t) = \sum_{\nu=0}^{\infty} \left(\frac{1}{\nu + \frac{1}{2} + \frac{\beta\omega_{\rm c} D_0}{2\pi}\Delta_\nu} - \frac{1}{\nu + \frac{1}{2}} \right) \tag{25}$$

where $t = T/T_{\rm c}$, $T_{\rm c}$ being the critical temperature of the pure superconductor, and we indicate by $D_0 = \frac{1}{3}\epsilon_{\rm F}\tau$ the diffusion constant with $\tau = \Gamma_0^{-1}$.

In (25) we introduced the notation:

$$\Delta_\nu = \frac{\Gamma_0}{\Gamma_\nu} = \frac{U}{\Pi_\nu^{\rm p-p}} \tag{26}$$

and we proceed to the evaluation of Δ_ν in the next section.

3. Vertex corrections and the self-consistent localization theory

It was pointed out by Vollhardt and Wölfle [11] that the particle–particle irreducible vertex function contains a class of diagrams that are obtained by inverting one line in the particle–hole reducible vertex part, like diagrams (a) and (b) in figure 3. If we consider only those diagrams we can write:

$$W_\nu^{\rm p-p}(\boldsymbol{r}_1\boldsymbol{r}_2 \mid \boldsymbol{r}_3\boldsymbol{r}_4) = \bar{W}_{\nu R}^{\rm p-h}(\boldsymbol{r}_1\boldsymbol{r}_2 \mid \boldsymbol{r}_4\boldsymbol{r}_3) \tag{27}$$

where $\bar{W}_{\nu\rm R}^{\rm p-h}$ contains all particle-hole diagrams that do not become particle–particle reducible by inverting one internal line, then diagrams (c) and (d) in figure 3 do not belong to this class. As in the present case particle-hole symmetry is lacking, we should consider also the corresponding equation for the particle–hole irreducible vertex part:

$$W_\nu^{\rm p-h}(\boldsymbol{r}_1\boldsymbol{r}_2 \mid \boldsymbol{r}_4\boldsymbol{r}_3) = W_{\nu\rm R}^{\rm p-p}(\boldsymbol{r}_1\boldsymbol{r}_2 \mid \boldsymbol{r}_3\boldsymbol{r}_4). \tag{28}$$

We should approximate as in (8)

$$W_\nu^{\rm p-h}(\boldsymbol{r}_1\boldsymbol{r}_2 \mid \boldsymbol{r}_4\boldsymbol{r}_3) \approx \Pi_\nu^{\rm p-h}\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\delta(\boldsymbol{r}_3 - \boldsymbol{r}_4)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_3) \tag{29}$$

which gives from figure 2(b):

$$W_{\nu R}^{p-h}(r_1 r_2 \mid r_4 r_3) \approx W_{\nu R}^{p-h}(r_1 - r_2)\delta(r_1 - r_3)\delta(r_2 - r_4) \tag{30}$$

with

$$W_{\nu R}^{p-h}(r - r') = U\delta(r - r') + \Pi_\nu^{p-h} \int d\rho\, G_\nu(r,\rho)G_{-\nu}(\rho,r)W_{\nu R}^{p-h}(\rho - r') \tag{31}$$

while from (9) and figure 2(a) we get:

$$W_{\nu R}^{p-p}(r; r') = U\delta(r - r') + \Pi_\nu^{p-p} \int d\rho\, G_\nu(r,\rho)G_{-\nu}(r,\rho)W_{\nu R}^{p-p}(\rho; r'). \tag{32}$$

The product of the two propagators going in opposite directions in (31) is translationally invariant even in the presence of a magnetic field, because the phase factors in (15) will cancel each other. This is a difference from the kernel in (32), where the phases of the propagators going in the same direction add. Equation (31) can then be solved by Fourier transformation with the result:

$$W_{\nu R}^{p-h}(k) = U + U\Pi_\nu^{p-h} \frac{Q_\nu(k)}{1 - \Pi_\nu^{p-h} Q_\nu(k)} \tag{33}$$

where we have from (15) and from [10]:

$$Q_\nu(k) = \int dr\, e^{ik\cdot r} g_\nu(r) g_{-\nu}(r) \approx \frac{1}{\Pi_\nu^{p-p}}(1 - \omega_\nu \tau \Delta_\nu - k^2 D_0 \tau \Delta_\nu^2). \tag{34}$$

The integral equation (32) appears in the calculation of the magnetoresistance and it was discussed in [10]. Proceeding in the same way; we first observe that the solution to (32) can be written exactly as:

$$W_{\nu R}^{p-p}(r, r') = e^{i\omega_c (x - x')(y + y')} \Lambda_\nu (r - r') \tag{35}$$

with $\Lambda(r)$ being the solution of the integral equation:

$$\Lambda(r) = U\delta(r) + \Pi_\nu^{p-p} \int d\rho\, e^{i\omega_c(\rho_x r_y - \rho_y r_x)} g_\nu(\rho) g_{-\nu}(\rho) \Lambda_\nu (r - \rho). \tag{36}$$

Equation (36) cannot be solved in closed form by Fourier transformation, but we obtain to lowest order [10] in ω_c:

$$\Lambda_\nu(k) \approx U \frac{[1 - \Pi_\nu^{p-p} Q_\nu(k)]}{[1 - \Pi_\nu^{p-p} Q_\nu(k)]^2 + (\omega_c D_0 \tau \Delta_\nu^2)^2} \tag{37}$$

with $Q_\nu(r)$ as given in (34). The result in (37) corresponds to the semiclassical approximation although it is not identical [10] to it.

By going back to real space we do not recover the δ-function behaviour implied in the self-consistent equations (27) and (28) together with (8) and (29) then we further approximate in (31) and (36):

$$W_{\nu R}^{p-h}(r - r') \approx \delta(r - r') \left(U + \ell_f^3 \int \frac{dk}{(2\pi)^3} \Pi_\nu^{p-h} Q_\nu(k) W_{\nu R}^{p-h}(k) \right) \tag{38}$$

$$\Lambda_\nu(r - r') \approx \delta(r - r') \left(U + \ell_f^3 \int \frac{dk}{(2\pi)^3} \Pi_\nu^{p-p} Q_\nu(k) \Lambda_\nu(k) \right) \tag{39}$$

where ℓ_f is of the order of the mean free path. We obtain by identifying the terms in brackets in (38) and (39) as Π_ν^{p-p} and Π_ν^{p-h}, respectively, the pair of coupled equations:

$$\Delta_\nu = 1 - r + r\frac{\pi}{2}\left(\frac{\omega_\nu \tau}{\Delta_\nu} + \frac{1}{\Delta_\nu^3}(\sigma_\nu - \Delta_\nu)\right)^{1/2} \tag{40}$$

$$\sigma_\nu = 1 - r + r\frac{\pi}{2}\,\mathrm{Re}\left(\frac{\omega_\nu \tau}{\Delta_\nu} + \mathrm{i}\omega_c \tau D_0\right)^{1/2} \tag{41}$$

where Δ_ν was introduced in (26) and we have defined:

$$\sigma_\nu = \frac{U}{\Pi_\nu^{p-h}}. \tag{42}$$

Also we have introduced:

$$r = \frac{1}{2\pi^2}(k_c \ell_f)^3 \tag{43}$$

with k_c being a standard cut-off [10]. In order to derive (40) and (41) from (38) and (39), we followed closely [9] in considering r to be a small parameter, although at the localization transition we make $r = 1$.

We notice that σ_ν in (41) corresponds to the magnetoresistance [10] while the upper critical field in (25) depends on Δ_ν. At zero field these two quantities coincide and we obtain that (40) and (41) reduce to Shapiro's result [9].

Equations (40) and (41) should also be compared with the work by Yoshioka *et al* [13], where a self-consistent localization theory in the presence of a magnetic field based on phenomenological semiclassical arguments, in two dimensions, is presented. We can identify their $\Phi(z)$, $\Psi(z)$, for $z = \mathrm{i}\omega_\nu$, with our σ_ν and Δ_ν, and while our (41) would correspond in three dimensions to their equation for $\Phi(z)$, equation (40) for Δ_ν is completely different from the equation for $\Psi(z)$ in [13], because they missed the shift in the diffusion pole due to having $\Pi_\nu^{p-p} \neq \Pi_\nu^{p-h}$. Their expression for $\Psi(z)$ would be obtained from (40) by setting $\Delta_\nu = \sigma_\nu$ in the RHS.

A closed solution of the problem would require us to solve for (40) and (41) for every frequency and field value, in order to solve for (25). We follow [8] by taking a more phenomenological approach and we find from (40) and (41) following the asymptotic solutions:

(a) strong field or weakly localized regime:

$$\frac{\omega_\nu \tau}{\omega_c \tau D_0} < \Delta_\nu < \sqrt{\frac{\sigma_\nu - \Delta_\nu}{\omega_\nu \tau}} \tag{44}$$

$$\begin{aligned} \Delta_\nu &\approx 1 - r + r\frac{\pi}{2}\sqrt{\omega_c \tau D_0} \\ \sigma_\nu &\approx \Delta_\nu + (1-r)r\frac{\pi}{2}\sqrt{\omega_c \tau D_0} \end{aligned} \tag{45}$$

(b) weak field or strongly localized regime:

$$\sqrt{\frac{\sigma_\nu - \Delta_\nu}{\omega_\nu \tau}} < \Delta_\nu < \frac{\omega_\nu \tau}{\omega_c \tau D_0} \tag{46}$$

$$\begin{aligned} \sigma_\nu &= \Delta_\nu + \mathrm{O}(\omega_c^2) \\ \Delta_\nu &= \tfrac{2}{3}(1-r) + (\omega_\nu \tau)^{1/3}. \end{aligned} \tag{47}$$

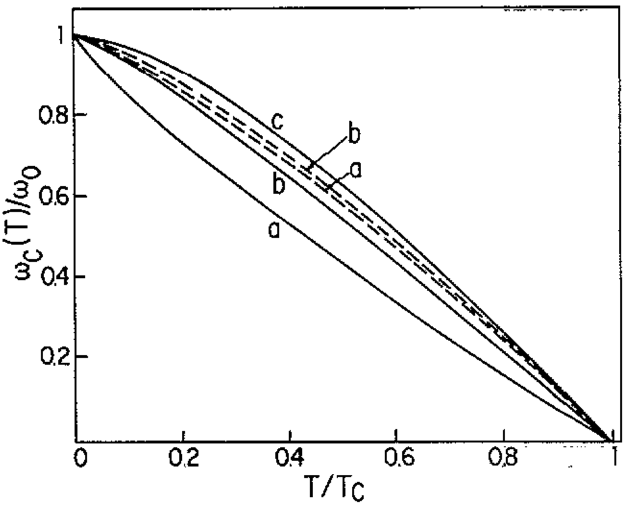

Figure 4. Normalized upper critical field ω_c/ω_o as a function of temperature for some values of the disorder parameter $C = \frac{2}{3}\pi(\tau T_c)^{-1/3}(1-r)$, which increases for decreasing disorder. The full curve is obtained from Δ_ν in (47). The broken curve corresponds to Δ_{KK} in (48). The labels (a), (b) and (c) are for $C = 0$, 0.5 and 5. For $C = 5$ the two curves are indistinguishable. The value $C = 0$ indicates the metal–insulator transition.

The scaling function in section II of [8] is an interpolation formula between these two regimes and it corresponds to

$$\Delta_{KK} = \tfrac{2}{3}(1-r) + (\omega_\nu\tau)^{1/3} + \alpha_2\sqrt{\omega_c\tau D_0} \tag{48}$$

with α_2 being a phenomenological constant, if we identity the localization length $\xi \approx (1-r)^{-1}$.

We obtain in fact by using (48) that the equation (25) may be recast in the form:

$$\ln(t) = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{\beta}{2\pi\tau}\delta^2(y^2 + \alpha_2 y^3)\right) + \int_0^\infty \mathrm{d}z \left(\frac{1}{1+z+\gamma z^{1/3}} - \frac{1}{1+z}\right) \tag{49}$$

where $\delta = \frac{2}{3}(1-r) \approx \xi^{-1}$, $y = \sqrt{\omega_c\tau D_0}/\delta$ and

$$\gamma(y) = \left(\frac{y}{1+\alpha_2 y}\right)^{2/3}. \tag{50}$$

Equation (49) is just the equation (55) for B_{c_2} in the paper by Kotliar and Kapitulnik [8]. It is important to remark here that their numerical calculations where performed by replacing $\gamma(y=\infty)$ in the RHS of (49), as is stated in section VI of [8], while our results were obtained directly from our (25).

4. Numerical results and conclusions

The results for the upper critical field B_{c_2} in figure 4 were obtained by introducing Δ_ν from (47) and (48) into (25).

We observe in figure 4 a striking change of curvature with increasing disorder ($r \to 1$) when we use (47), due to the $(\omega_\nu \tau)^{1/3}$ term. When we use (48) with $\alpha_2 = 1$, the importance of this term is reduced and the curvature remains negative. Then in our formulation the change in curvature is obtained by using (47), which is equivalent to introducing $\Delta_\nu(\omega_c = 0)$ into (25) and to considering the self-consistent equation to lowest order in ω_c. The theory of Kotliar and Kapitulnik [8] would correspond to (48) recast in the form of (49) with the further approximation $\gamma(y) \approx \gamma(\infty)$.

To conclude, we present a study of the upper critical field of disordered superconductors in terms of the Landau levels that avoids the use of the semiclassical approximation from the start. We go beyond the classical theory of Werthamer, Helfand and Hohenberg [1] by including higher-order vertex corrections in the particle–particle propagator that forms the kernel of the integral equation for the gap function.

We consider only those diagrams that represent the processes responsible for the localization transition and we generalize the self-consistent theory of Vollhardt and Wölfle [11] to the presence of a magnetic field, by using the exact electron eigenstates.

This is the main difference between our formulation and the previous work in [8] and [13] for the upper critical field and the magnetoconductance, respectively. In these papers all the equations are worked out for $B = 0$ by exploiting translational and time reversal invariance, while the magnetic field is switched on phenomenologically afterwards.

By working from the start within a formalism that lacks particle–hole symmetry we have the advantage of identifying all the terms that would vanish in the symmetric case. This permitted us to find a suppression of the diffusion pole of (40) that was missing in [13].

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